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Variational treatment of quasi-stationary thermal conduction through flat plates

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Abstract—This paper deals with the application of variational principles in the study of heat conduction through the flat plate in the quasi-stationary regime of the heating agent temperature variation; the study deals with the variable regime during the period preceding the quasi-stationary heating generalization throughout the plate. The new solution to the problem is expressed by two relations. The author has contributed to the tackling of this problem.

1. INTRODUCTION

IN QUASI-stationary thermal conduction the temperature of the hot medium increases with a constant velocity. If an infinite value of the heat transfer coefficient is admitted, the heated surface of the plate takes the temperature of the medium, while heating of the insulated surface is delayed. At the beginning, a variable temperature difference appears between the two sides of the plate; after some time, the temperature increases uniformly in all points of the plate, while the temperature difference between the two surfaces has a maximum value and is constant.

This paper presents a variational treatment of quasi-stationary heat transfer throughout a plate, based on the concept of a thermal potential, dissipation function and generalized thermal force.

Based on the author's investigations, a computation relation of the time required for heat to penetrate through the flat plate is proposed; a computation relation of the temperature on the insulated surface of the plate is also proposed. The results are rendered by two simple physical relations, that enable a fast evaluation of the implications of this temperature variation on the surfaces of a plate.

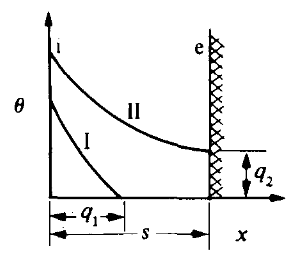

FIG. 1. Temperature distribution in a flat plate quasi-stationary heated and insulated at $x = s$.

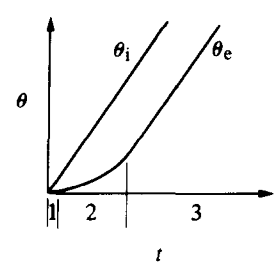

FIG. 2. Phases of the heating process and surface temperature variations.

2. HEAT PENETRATION IN THE PLATE

Consider a plate of thickness s with constant thermal conductivity λ, specific heat c, and density ρ. The plate is initially at temperature $\theta = 0$. One surface of the plate (i), located at $x = 0$, is quasi-stationary heated at a velocity v; the other surface (e) at $x = s$ is thermally insulated. The temperature distribution is shown in Fig. 1.

The heating process is divided into three phases as shown in Fig. 2. In the first phase (1) it is assumed that the heat has penetrated to a depth $x = q_1$, smaller than the thickness s and that the temperature distribution is well approximated by the expression

$$\theta = vt\left(1 - \frac{x}{q_1}\right)^2. \tag{1}$$

This parabolic approximation is shown by curve (I) in Fig. 1. The penetration depth q_1 is a generalized coordinate to be determined as a function of time. Since this is a one-dimensional problem it is sufficient to consider a cylinder of unit cross-section of axis perpendicular to the wall.

2.1. *Dimensional analysis*

For the variational treatment of heat transfer through the flat plate it is necessary to know the relation between the descriptive parameters $q_1 = f(t, \lambda, c, \rho, vt)$; the form of this relation can be established by dimensional analysis. The relation can be written in the form

NOMENCLATURE

a	thermal diffusivity, $\lambda/c\rho$
b, d, e, h, j	physical quantity exponents
c	specific heat
D	dissipation function
H	heat displacement
$\dot{H}$	local rate of heat flow per unit area, $(\partial H/\partial q)\cdot(\partial q_1/\partial t)$; $(\partial H/\partial q_2)\cdot(\partial q_2/\partial t)$
k	non-dimensional numerical factor
L, T, M, Q	fundamental physical quantities
q_1	depth
$\dot{q}_1$	$\partial q_1/\partial t$
q_2	temperature
$\dot{q}_2$	$\partial q_2/\partial t$
Q_1, Q_2	generalized thermal force
r	numerical factor
s	thickness
S	surface
t	time
v	velocity
V	thermal potential
x	coordinate.

Greek symbols

θ	temperature
Θ	fundamental physical quantity
λ	thermal conductivity
ρ	density.

Subscripts

e	external
i	internal.

$$q_1 = kt^b\lambda^d c^e \rho^h (vt)^j \tag{2}$$

where the physical quantity exponents $b, d, e, \ldots$, and the non-dimensional factor k are unknown. The variable dimensional matrix for five fundamental quantities is

	q_1	t	λ	c	ρ	vt
L	1	0	−1	0	−3	0
T	0	1	−1	0	0	0
M	0	0	0	−1	1	0
Θ	0	0	−1	−1	0	1
Q	0	0	1	1	0	0

(3)

The dimensional equations given by matrix (3) should be introduced in equation (2). The dimensional homogeneity condition of equation (2) is expressed by

$$\begin{aligned} -d \quad -3h &= 1 \quad (L)\\ b-d &= 0 \quad (T)\\ -e+h &= 0 \quad (M)\\ -d-e+j &= 0 \quad (\Theta)\\ d+e &= 0 \quad (Q) \end{aligned} \tag{4}$$

The solution of this determinate system of equations is

$$b = d = \frac{1}{2}, \qquad c = h = -\frac{1}{2}, \qquad j = 0. \tag{5}$$

Equation (2) takes the form

$$q_1 = k\sqrt{(at)} \tag{6}$$

since $a = \lambda/c\rho$.

Now, equation (1) becomes

$$\theta = k\frac{q_1^2}{a}v\left(1-\frac{x}{q_1}\right)^2. \tag{7}$$

With the value of equation (7), the thermal potential is

$$V = \frac{1}{2}c\rho\int_0^{q_1}\theta^2\,dx = \frac{1}{10}k^2\frac{c\rho}{a^2}v^2q_1^5. \tag{8}$$

The heat displacement H is derived from the temperature θ by using the law of energy conservation $c\rho\theta = -\operatorname{div} H$, which in this case becomes

$$c\rho\theta = -\frac{dH}{dx}. \tag{9}$$

By taking into account the condition $H = 0$ at $x = q_1$, we obtain

$$H = k\frac{c\rho}{a}v\left(\frac{1}{3}q_1^3 - q_1^2x + q_1x^2 - \frac{1}{3}x^3\right). \tag{10}$$

The dissipation function is

$$D = \frac{1}{2\lambda}\int_0^{q_1}\dot{H}^2\,dx = \frac{1}{10}k^2\frac{c\rho}{a^3}v^2q_1^5\dot{q}_1^2 \tag{11}$$

where the vector takes the expression

$$\dot{H} = \frac{\partial H}{\partial q_1}\cdot\frac{\partial q_1}{\partial t} = \frac{\partial H}{\partial q_1}\cdot\dot{q}_1.$$

The generalized thermal force Q_1 is obtained by considering the virtual heat displacement

$$\delta H = k\frac{c\rho}{a}vq_1^2\delta q_1$$

at $x = 0$. In conformity with the variational method, we can write

$$Q_1\delta q_1 = vt\delta H \tag{12}$$

hence

$$Q_1 = k^2\frac{c\rho}{a^2}v^2q_1^4. \tag{13}$$

The equation for the unknown Lagrangian coordinate q_1 is

$$\frac{\partial V}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} = Q_1. \tag{14}$$

Substitution of equations (8), (11) and (13) yields

$$\frac{1}{5} q_1 \dot{q}_1 = \frac{1}{2} a. \tag{15}$$

This is a first-order equation with the time differential q_1. With the initial condition $q_1 = 0$ at $t = 0$, we find

$$q_1^2 = 5at. \tag{16}$$

The first phase ends when $q_1 = s$ at a time t equal to

$$t_1 = 0.2 \frac{s^2}{a}. \tag{17}$$

This transit time measures the period required for heat to penetrate through a thickness s of a given material.

3. TEMPERATURE VARIATION AT THE INSULATED SURFACE

In the second phase (2), corresponding to time $t > t_1$, the temperature rises at the insulated boundary $x = s$. The temperature in this phase is also assumed to be well represented by a parabolic approximation

$$\theta = (vt - vt_1 - q_2)\left(1 - \frac{x}{s}\right)^2 + q_2. \tag{18}$$

This is illustrated by curve (II) in Fig. 1. The generalized coordinate q_2 is the unknown temperature at the boundary $x = s$. With the value of θ from equation (18), the thermal potential is

$$V = \frac{1}{2} c\rho \int_0^s \theta^2 \, dx = \frac{1}{30} c\rho s [3v^2(t^2 - t_1^2) + 4vq_2(t - t_1) - 6v^2 t t_1 + 8q_2^2]. \tag{19}$$

The heat displacement H is obtained by integrating $c\rho\theta = -\partial H/\partial x$; assuming for θ the value given by equation (18) and the boundary condition $H = 0$ at $x = s$, to give

$$H = c\rho\left[(vt - vt_1 - q_2)\frac{(s - x)^3}{3s^2} + q_2(s - x)\right]. \tag{20}$$

In this case, the vector H takes the form

$$\dot{H} = \frac{\partial H}{\partial q_2} \cdot \frac{\partial q_2}{\partial t} = \frac{\partial H}{\partial q_2} \cdot \dot{q}_2.$$

The dissipation function is

$$D = \frac{1}{2\lambda} \int_0^s \dot{H}^2 \, dx = \frac{34}{315} \frac{c^2 \rho^2}{a} s^3 \dot{q}_2^2. \tag{21}$$

The generalized thermal force Q_2 is obtained by considering the virtual heat displacement $\delta H = (2/3)c\rho s \delta q_2$ at $x = 0$. In conformity with the variational method

$$Q_2 \delta q_2 = v(t - t_1)\delta H \tag{22}$$

hence

$$Q_2 = \frac{2}{3} c\rho v s (t - t_1). \tag{23}$$

By introducing equations (19), (21) and (23) in the Lagrangian equation

$$\frac{\partial V}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} = Q_2 \tag{24}$$

is obtained the following linear differential equation for q_2:

$$q_2 + \frac{17}{42} \frac{t_1}{0.2} \dot{q}_2 - v(t - t_1) = 0 \tag{25}$$

where t_1 is the transit time, equation (17). By integrating equation (25) with the initial value $q_2 = 0$ for $t = t_1$, one obtains

$$q_2 = 2vt_1 \exp 0.5\left(1 - \frac{t}{t_1}\right) + vt - 3vt_1. \tag{26}$$

4. QUASI-STATIONARY RISE OF THE TEMPERATURE

In the third phase (3), the temperature will increase uniformly at all points of the flat plate with velocity v; the temperature difference $\theta_i - \theta_e$ will take the maximum value.

The temperature field in the plate is described by the Fourier differential equation which, in the absence of the inner heat sources, has the form

$$a \frac{\partial^2 \theta}{\partial x^2} = v \tag{27}$$

written in Cartesian coordinates and considering one-dimensional heat transmission; its general solution is known and has the form

$$\theta = \frac{v}{2a} x^2 + k_1 x + k_2. \tag{28}$$

To determine the constants k_1 and k_2, two conditions will be necessary; $\theta = \theta_i$ at $x = 0$ and constant $k_2 = \theta_i$; the heat passing throughout surface i during the time unit leads to the uniform increase of the plate temperature; therefore

$$-\lambda S\left(\frac{\partial \theta}{\partial x}\right)_{x=0} = c\rho s S \frac{\partial \theta}{\partial t}; \tag{29}$$

the constant $k_1 = -sv/a$. Hence, the temperatures of the two surfaces will have the values

$$\theta_i = vt, \qquad \theta_e = vt - \frac{vs^2}{2a}. \tag{30}$$

5. CONNECTION OF THE TEMPERATURES OF PHASES (2) AND (3)

At the moment t_2, the temperature difference $\theta_i - q_2$ from the end of phase (2) has the maximum value

$$(\theta_i - q_2)_{\max} = 3vt_1 = 0.6\frac{vs^2}{s} \tag{31}$$

considering equation (17). This difference must be equal to the constant temperature difference from phase (3)

$$\theta_i - \theta_e = 0.5\frac{vs^2}{a}. \tag{32}$$

A distinction exists between equations (31) and (32) expressed by the ratio $r = 0.5/0.6$; equations (17) and (26) must be modified; thus

$$t_1^m = 0.167\frac{s^2}{a} \tag{33}$$

i.e.

$$q_2^m = vt - rvt_1\left[3 - 2\exp 0.5\left(1 - \frac{t}{t_1}\right)\right]. \tag{34}$$

Time t_2 is obtained from the relation $\theta_i - \theta_e = \theta_i - q_2^m$ which has the form

$$\frac{vs^2}{2a} = rvt_1\left[3 - 2\exp 0.5\left(1 - \frac{t_2}{t_1}\right)\right] \tag{35}$$

and admits the solution $t_2 = \infty$.

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APPENDIX

Application

As an application, one may consider a steel plate with a thermal diffusivity $a = 3.42\,\text{em}^2\,\text{min}^{-1}$; the choice is based on the fact that this material is widely used in the building of machines subjected to high temperatures; the thickness of the flat plate is $s = 16\,\text{cm}$ and the velocity $v = 3°\text{C}\,\text{min}^{-1}$.

On the basis of the results of equation (17) $t_1 = 15\,\text{min}$; the temperature θ_i is given by equation $(30)_1$; from equation (34) the temperature q_2^m may be calculated. In Table A1, the computed values of the temperatures are presented, for different values $t > t_1 = 15\,\text{min}$.

Table A1. Temperature values in a flat plate

t (min)	θ_i (°C)	q_2^m (°C)	$\theta_i - q_2^m$ (°C)
25	75	16.24	58.76
45	135	50.09	84.91
65	195	96.66	98.34
95	285	177.71	107.29
125	375	264.42	110.58
225	675	562.57	112.33
425	1275	1162.50	112.36

The numerical calculus shows that phase (2) practically ends after a finite time, when the temperature difference $\theta_i - q_2^m$ takes the maximum value $vs^2/2a = 112.3°\text{C}$ from phase (3).

ETUDE VARIATIONNELLE DE LA CONDUCTION THERMIQUE QUASISTATIONNAIRE DANS DES PLAQUES PLANES

Résumé—On présente l'application des principes variationnele à l'étude de la conduction de la chaleur dans la plaque plane, quand la température du fluide chauffant varie en régime quasistationnaire; l'étude concerne le régime variable qui précède l'échauffement quasistationnaire de la plaque entière. La nouvelle résolution de ce problème conduit à des relations physiques simples. Quelques contributions originales de l'auteur, ont été utilisées à la résolution du problème.

VARIATIONELLES STUDIUM DER QUASISTATIONÄREN WÄRMELEITUNG DURCH DIE FLACHEN PLATTEN

Zusammenfassung—Es wird die Anwendung variationäller Prinzipien beim Studium der Wärmeleitung durch die flache Platte, im Falle quasistationärem Temperaturschwankungsregim der erwärmenden Flüssigkeit, dargestellt; das Studium bezieht sich auf das variable Regime welches der Verallgemeinerung der quasistationären Erwärmung in allen Punkten der Platte vorangeht. Die neue Lösung dieser Frage konkretisiert sich durch einfache Berechnungsgleichungen. Bei der Behandlung dieses Problems hatte such der Verfasser eine persönlichen Beitrag.

ПРИМЕНЕНИЕ ВАРИАЦИОННЫХ ПРИНЦИПОВ ДЛЯ ОПИСАНИЯ КВАЗИСТАЦИОНАРНОГО ПЕРЕНОСА ТЕПЛА ЧЕРЕЗ ПЛОСКИЕ ПЛАСТИНЫ

Аннотация—Рассматривается применение вариационных принципов для описания переноса тепла через плоскую пластину при квазистационарном режиме изменения температуры теплового агента. Исследуется режим, предшествующий установлению квазистационарного нагрева. Получено новое решение задачи, представленное в виде двух соотношений.